Supplement: Reconstruction equation for solitons in the Korteveg-de Vries equation

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We show that the reconstruction equation given in [2] correctly recovers the speed of solitons in the Korteveg-de Vries equation (KdV), regardless of the choice of template used for template fitting.

Solitons in KdV

The Korteveg-de Vries equation is given by

$$u_t + 6uu_x + u_{xxx} = 0 \quad (1)$$

(see, e.g., [1]) and admits soliton solutions of the form $u = \varphi(x - ct)$. Substituting into (1), we see these solutions must satisfy

$$-c\varphi' + 6\varphi\varphi' + \varphi''' = 0. \quad (2)$$

The method of [2] expresses a shifted version of the solution as

$$\hat{u}(x,t) = u(x - \alpha(t), t) \quad (3)$$

where the shift amount $\alpha(t)$ satisfies a reconstruction equation

$$\dot{\alpha} = \langle D(\hat{u}), u'_0 \rangle \langle \hat{u}_x, u'_0 \rangle, \quad (4)$$

where $D(u) = -6uu_x - u_{xxx}$, and $u_0(x)$ is a template function, chosen arbitrarily. We wish to see under what conditions the reconstruction equation correctly recovers the shift amount $\alpha(t) = -ct$, so that $u(x,t) = \hat{u}(x - ct, t)$ and $\hat{u}(x,t) = \varphi(x)$.

For the soliton solution $u = \varphi(x - ct)$, we have

$$\hat{u}(x,t) = u(x - \alpha(t), t) = \varphi(x - \alpha(t) - ct) = S_{\alpha+ct}[\varphi](x) \quad (5)$$
where $S_u[v](x) = v(x - a)$ is the shift operator on periodic functions. Hence

\[
D(\dot{u}) = -6\dot{u}\dot{u}_x - \dddot{u}_x \\
= S_{a+ct}[-6\varphi' - \varphi'''] \\
= S_{a+ct}[-c\varphi']
\]

by (2), so the reconstruction equation gives

\[
\dot{\alpha} = \frac{\langle-cS_{a+ct}[\varphi'], u'_0\rangle}{\langle S_{a+ct}[\varphi'], u'_0\rangle} = -c 
\]

(6)

as long as $\langle S_{a+ct}[,\varphi'], u'_0\rangle \neq 0$. Hence $\alpha(t) = -ct$, independent of the template $u_0$, as desired.

**General traveling waves**

More generally, if a PDE given by $\dot{u} = D(u)$ admits traveling wave solutions $u = \varphi(x - f(t))$, and $D$ is equivariant under translations, then $\varphi$ must satisfy

\[
-f'\varphi' = D(\varphi). 
\]

(7)

Again defining $\tilde{u}(x, t) = u(x - \alpha(t)) = S_{a+f}[\varphi](x)$, the reconstruction equation becomes

\[
\dot{\alpha} = \frac{\langle D(\tilde{u}), u'_0\rangle}{\langle \dot{u}_x, u'_0\rangle} = \frac{\langle S_{a+f}[D(\varphi)], u'_0\rangle}{\langle S_{a+f}[\varphi'], u'_0\rangle} = \frac{\langle S_{a+f}[-f\varphi'], u'_0\rangle}{\langle S_{a+f}[\varphi'], u'_0\rangle} = -f, 
\]

(8)

where the second equality holds by equivariance of $D$, and the third by (7). Integrating, we have $\alpha(t) = -f(t)$, once again recovering the correct position of the traveling wave.

**References**
