Supplement: Reconstruction equation for solitons in the Korteveg-de Vries equation

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We show that the reconstruction equation given in [2] correctly recovers the speed of solitons in the Korteveg-de Vries equation (KdV), regardless of the choice of template used for template fitting.

Solitons in KdV

The Korteveg-de Vries equation is given by

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

(see, e.g., [1]) and admits soliton solutions of the form $u = \varphi(x - ct)$. Substituting into (1), we see these solutions must satisfy

$$-c\varphi' + 6\varphi\varphi' + \varphi''' = 0. \tag{2}$$

The method of [2] expresses a shifted version of the solution as

$$\hat{u}(x,t) = u(x - \alpha(t), t) \tag{3}$$

where the shift amount $\alpha(t)$ satisfies a ***reconstruction equation***

$$\dot{\alpha} = \frac{\langle D(\hat{u}), u_0' \rangle}{\langle \hat{u}_x, u_0' \rangle}, \tag{4}$$

where $D(u) = -6uu_x - u_{xxx}$, and $u_0(x)$ is a template function, chosen arbitrarily. We wish to see under what conditions the reconstruction equation correctly recovers the shift amount $\alpha(t) = -ct$, so that $u(x,t) = \hat{u}(x - ct, t)$ and $\hat{u}(x,t) = \varphi(x)$.

For the soliton solution $u = \varphi(x - ct)$, we have

$$\hat{u}(x,t) = u(x - \alpha(t), t) = \varphi(x - \alpha(t) - ct) = S_{\alpha+ct}[\varphi](x) \tag{5}$$

where $S_a[v](x) = v(x-a)$ is the shift operator on periodic functions. Hence

$$\begin{aligned} D(\hat{u}) &= -6\hat{u}\hat{u}_x - \hat{u}_{xxx} \\ &= S_{\alpha+ct}[-6\varphi\varphi' - \varphi'''] \\ &= S_{\alpha+ct}[-c\varphi'] \end{aligned}$$

by (2), so the reconstruction equation gives

$$\dot{\alpha} = \frac{\langle -cS_{\alpha+ct}[\varphi'], u_0' \rangle}{\langle S_{\alpha+ct}[\varphi'], u_0' \rangle} = -c \tag{6}$$

as long as $\langle S_{\alpha+ct}[\varphi'], u_0' \rangle \neq 0$. Hence $\alpha(t) = -ct$, independent of the template u_0, as desired.

General traveling waves

More generally, if a PDE given by $\dot{u} = D(u)$ admits traveling wave solutions $u = \varphi(x - f(t))$, and D is equivariant under translations, then φ must satisfy

$$-\dot{f}\varphi' = D(\varphi). \tag{7}$$

Again defining $\hat{u}(x,t) = u(x - \alpha(t)) = S_{\alpha+f}[\varphi](x)$, the reconstruction equation becomes

$$\dot{\alpha} = \frac{\langle D(\hat{u}), u_0' \rangle}{\langle \hat{u}_x, u_0' \rangle} = \frac{\langle S_{\alpha+f}[D(\varphi)], u_0' \rangle}{\langle S_{\alpha+f}[\varphi'], u_0' \rangle} = \frac{\left\langle S_{\alpha+f}[-\dot{f}\varphi'], u_0' \right\rangle}{\langle S_{\alpha+f}[\varphi'], u_0' \rangle} = -\dot{f}, \tag{8}$$

where the second equality holds by equivariance of D, and the third by (7). Integrating, we have $\alpha(t) = -f(t)$, once again recovering the correct position of the traveling wave.

References

[1] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry.* Number 17 in Texts in Applied Mathematics. Springer, second edition, 1994.

[2] C. W. Rowley and J. E. Marsden. Reconstruction equations and the Karhunen-Loève expansion for systems with symmetry. *Phys. D*, 142:1–19, 2000.